Joseph Fourier and the Accusations of Insufficient Rigor

David Bakker — Utrecht University
35th Novembertagung, Nancy
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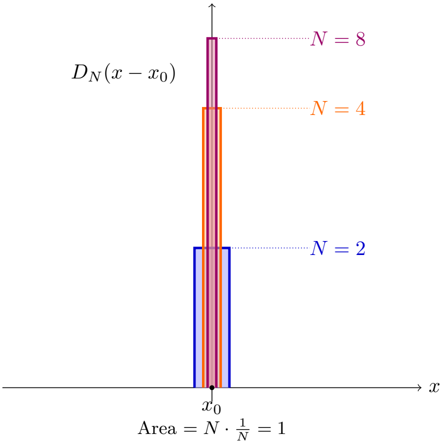
Laurent Schwartz

- Nancy-Université 1945–1953
- Distribution: *generalized* function
- Points vs. fuzzy points
- Dirac Delta distribution



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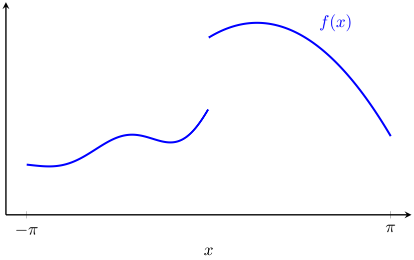
- Q: When can we do so? Fourier: always
- Proof: via Fourier inversion Theorem



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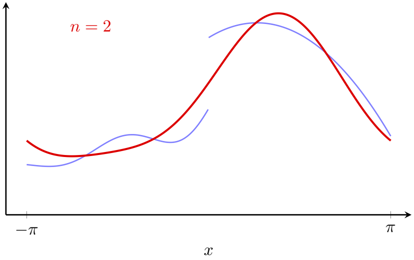

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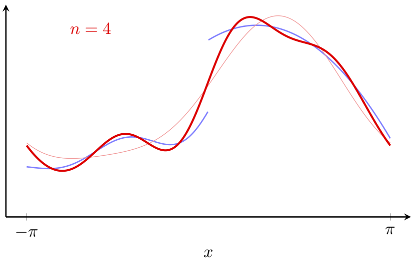

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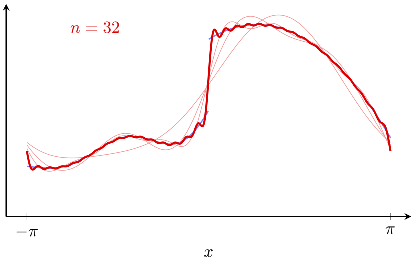

Very briefly: Context for Fourier's proof

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{j} a_k \cos(kx) + b_k \sin(kx), \quad j \to \infty$$

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$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos(k\alpha)d\alpha, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin(k\alpha)d\alpha$$

So that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)d\alpha + \sum_{k=1}^{j}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos(k\alpha)d\alpha\right]\cos(kx) + \left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin(k\alpha)d\alpha\right]\sin(kx)$$

Interchange:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\left[\frac{1}{2} + \sum_{k=1}^{j}\cos(k\alpha)\cos(kx) + \sin(k\alpha)\sin(kx)\right]d\alpha$$

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=&\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)D_j(\alpha-x)d\alpha, \qquad\qquad j\to\infty
\end{aligned}
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where

$$
D_j(r)=\cos jr+\sin jr\,\frac{\sin r}{1-\cos r}
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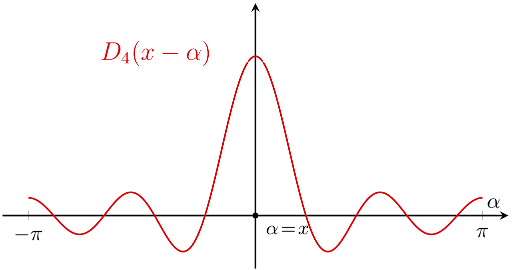
Fourier's Proof

We want to prove:

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha) D_j(x-\alpha)\, d\alpha, \qquad j \to \infty$$

To show: $f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)D_j(x-\alpha)d\alpha, \quad j \to \infty$

Let's look at $D_j(x-\alpha)$



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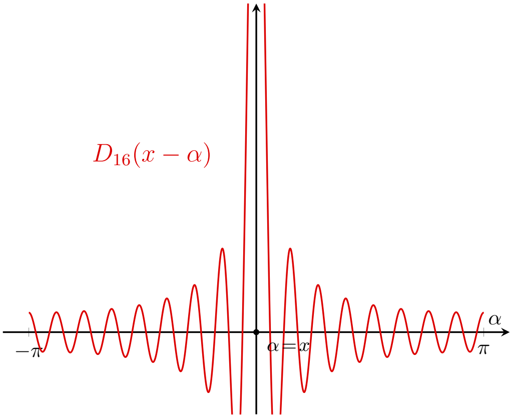

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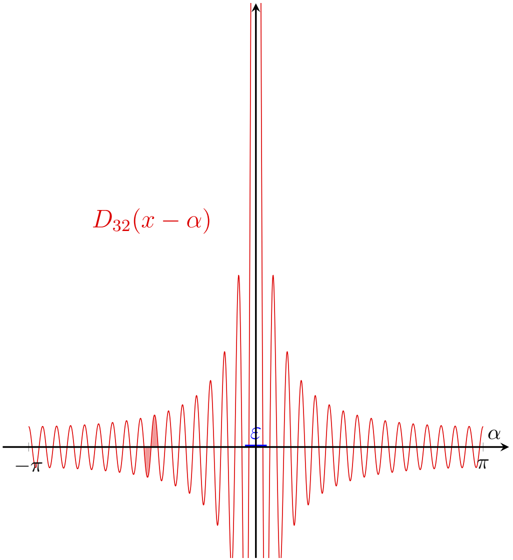

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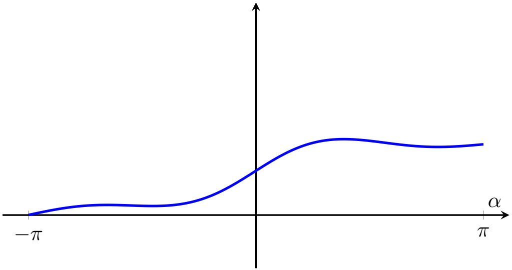
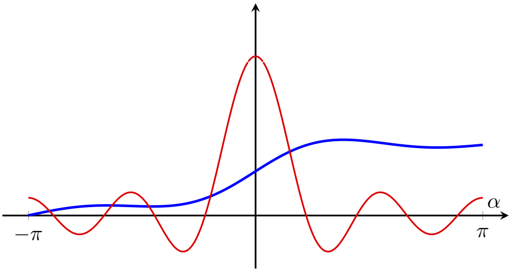
Now construct $f(\alpha)$



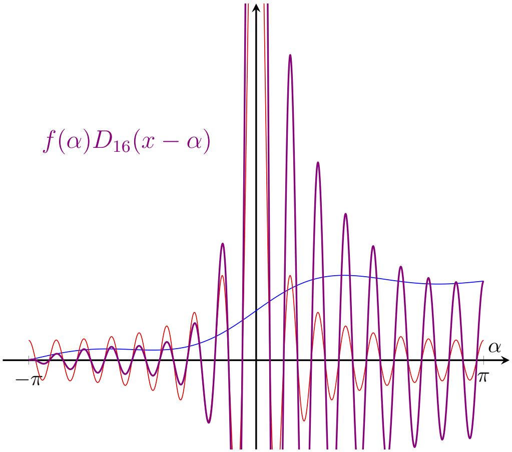
To show: $f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)D_j(x-\alpha)d\alpha, \quad j \to \infty$

Add $D_j(x-\alpha)$



To show: $f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha) D_j(x-\alpha)\,d\alpha, \quad j \to \infty$

Multiply $f(x)$ and $D_j(x-\alpha)$



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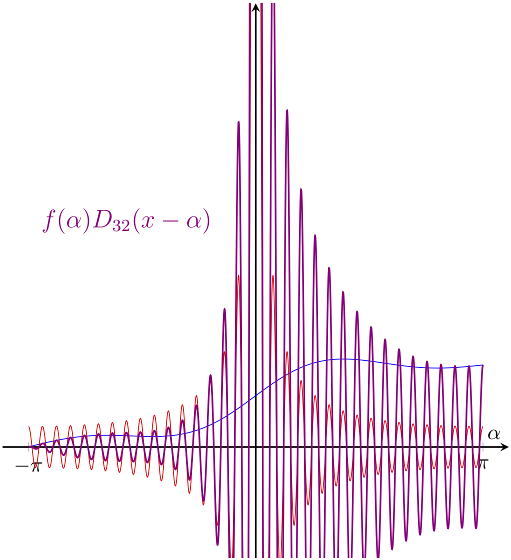

Is this proof any good?

Reactions

- Clifford Truesdell (1980):

could find nothing new in "FOURIER's theorem" except the sweeping generality of its statement and the preposterous legerdemain advanced as a proof.

- Ivor Grattan-Guinness (1990):

$$\int_0 (\sin x)/x \, dx = \pi \quad (1161.8)$$

was composed of a sequence of areas of (allegedly) equal and opposite magnitudes for large p, except around $x = 0$, whence the value was obtained. (1161.7) could then be obtained by a similar fudge (arts. 415–416) for "any" function $f(x)$, which was defined as 'a succession of values or ordinates each of which is arbitrary' and which 'succeed each other in any manner whatever' (art. 417).

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Fourier, on the other hand, consciously avoided implying that functions were analytical expressions. Yet in his "proof" of the convergence of Fourier series of an "arbitrary" function f, he explicitly used the fact that when "α differs infinitely

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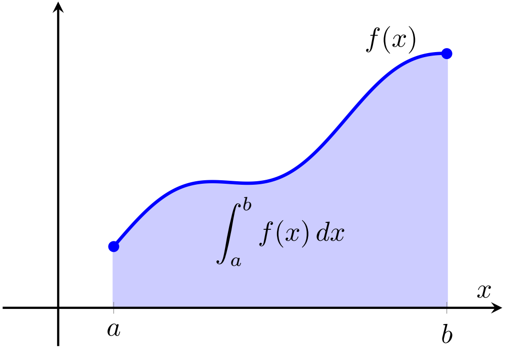
$$\int_{-\pi-x}^{\pi-x} f(x+r) \sin jr \frac{\sin r}{\sin V r} dr.$$

- **Gaston Darboux (in Fourier 1888):**

Fourier montre ici, par des raisonnements exacts dans le fond, bien que manquant de précision, que, lorsque le nombre j croît indéfiniment, cette intégrale « n'a de valeurs sub-

What is lacking in the proof?

- Not all details are worked out
- But, this is *insufficient* to explain the reactions
- Fourier's *geometrical* perspective: integral as area



given functions did not necessarily make sense. Therefore he focused on the definite integral $\int_a^b f(x)dx$ (putting the limits of integration at the top and bottom of the integral sign is in fact Fourier's idea) and stressed that it meant the area between the curve and the axis (Fourier 1822, §229).

Cauchy followed Fourier when he focused on the definite integral, but instead of relying on a vague notion of area, Cauchy defined the definite integral as the

(Lützen 2003)

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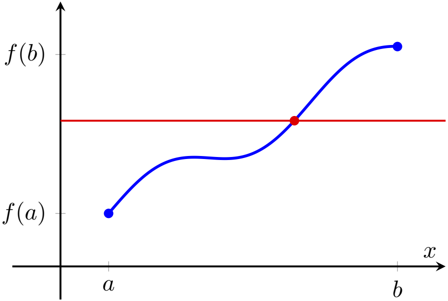
- Cauchy and the Intermediate Value Theorem



The Rigorization of Analysis

Cauchy:

Quant aux méthodes, j'ai cherché à leur donner toute la rigueur qu'on exige en géométrie, de manière à ne jamais recourir aux raisons tirées de la généralité de l'algèbre. Les raisons de cette espèce, quoi-

(Cauchy 1821)

- Two essentially different developments:
  1. Concern with the scope of generalizations
  2. Attack on geometrical ideas and forms of reasoning

The Rigorization of Analysis

Fourier:

this kind are very general, and susceptible of very different forms. We cannot delay over these developments, but it was necessary to exhibit the employment of geometrical constructions; for they solve without any doubt questions which may arise on the extreme values, and on singular values; they would not have served to discover these theorems, but they prove them and guide all their applications.

(Fourier 1822, tr. Freeman)

- Two essentially different developments:
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The Rigorization of Analysis

W.R. Hamilton:

before I had escaped from the camp of the old divergents. The principal theories of algebraical analysis (under which I include Calculi) require to be entirely remodelled, and Cauchy has done much already for this great object. Poisson also has done much, but he does not seem to me to have nearly so logical a mind as Cauchy, great as his talents and his clearness are: and both are in my judgment very far inferior to Fourier, whom I place at the head of the French School of Mathematical Philosophy, even above Lagrange and Laplace, though I rank *their* talents above those of Cauchy and Poisson. I must tell you I have been delighted with

(Graves 1885)

- Two essentially different developments:
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References

Cauchy (1821). *Cours d'Analyse de l'École royale polytechnique*

Grattan-Guinness (1990). *Convolutions in French Mathematics*.

Truesdell (1980). *The Tragicomical History of Thermodynamics*.

Lützen (2003). In Jahnke (ed.), *A History of Analysis*, ch. 6.

Fourier (1888). *Œuvres*, vol. I (ed. Darboux).

Graves (1885). *Life of Sir William Rowan Hamilton*, vol. II.

Schwartz (1997). *Un mathématicien aux prises avec le siècle.*

Vretblad (2003). *Fourier Analysis and Its Applications.*

For Cauchy on the Intermediate Value Theorem also see:

Barany (2013). *Stuck in the Middle: Cauchy's Intermediate Value Theorem and the History of Analytic Rigor*.

Bonus slides

Distributions, formally

Fuzzy points:

$$\mathcal{D}(U) := C_c^\infty(U)$$

Distributions:

$$\mathcal{D}'(U) \text{ all linear functionals on } \mathcal{D}(U) \text{ i.e., } \{D : C_0^\infty \to \mathbb{R} \mid D \text{ is linear}\}$$

Any function $f \in L^p(U)$ can be turned into a distribution D_f by the following procedure:

$$\varphi \mapsto \int_U f\varphi, \qquad \varphi \in C_c^\infty(U)$$

Function, or distribution?

Darboux also inserted:

ayant une somme indéterminée, on ne peut attacher aucun sens à l'expression

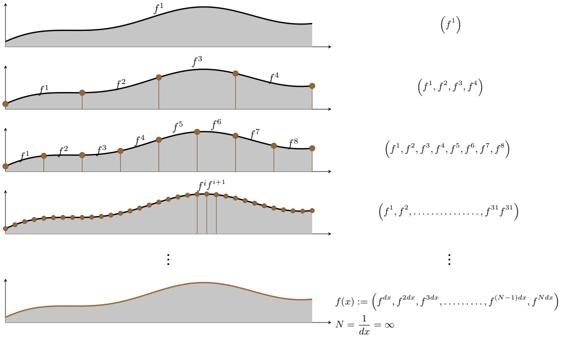
$$\frac{1}{2} + \sum_{i=1}^{i=\infty} \cos i(x - \alpha)$$

considérée par Fourier. G. D.

(Fourier 1888)

Function, or distribution?

Fourier's 'function':



En général, la fonction $f(x)$ représente une suite de valeurs, ou ordonnées, dont chacune est arbitraire. L'abscisse x pouvant recevoir une infinité de valeurs, il y a un pareil nombre d'ordonnées $f(x)$. Toutes ont des valeurs numériques *actuelles*, ou positives, ou négatives, ou nulles. On ne suppose point que ces ordonnées soient assujetties à une loi commune; elles se succèdent d'une manière quelconque, et chacune d'elles est donnée comme le serait une seule quantité.

(Fourier 1888)

Countably or uncountably infinite? $\mathbb{Q}$ or $\mathbb{R}$?

Pour résoudre clairement ces questions, il suffit de remonter aux principes qui servent de fondement à notre analyse.

Nous divisons l'intervalle X en un nombre infini n de parties égales à dx, en sorte que l'on a

$$n\,dx = X;$$

et, écrivant $f(x)$ au lieu de $x F(x)$, nous désignons par

$$f_1, \; f_2, \; f_3, \; \ldots, \; f_i, \; \ldots, \; f_n$$

les valeurs de $f(x)$ qui répondent aux valeurs

$$dx, \; 2\,dx, \; 3\,dx, \; \ldots, \; i\,dx, \; \ldots, \; n\,dx$$

(Fourier 1888)

Fourier, on the other hand, consciously avoided implying that functions were analytical expressions. Yet in his "proof" of the convergence of Fourier series of an "arbitrary" function f, he explicitly used the fact that when "α differs infinitely little from x, the value of $f\alpha$ coincides with fx" (Fourier 1822, §423); i.e., he assumed that any function is continuous in the modern sense. Cauchy and Fourier

(Lützen 2003)

I don't think this is what he assumes

Distributions and general derivatives

quelconque,

$$\frac{d^{2i} f(x)}{dx^{2i}} = \pm \frac{1}{2\pi} \int_a^b f(\alpha)\, d\alpha \int_{-\infty}^{+\infty} p^{2i} \cos p(x-\alpha)\, dp.$$

On écrit le signe supérieur lorsque i est pair, et le signe inférieur lorsque i est impair. On aura, en suivant cette même règle relative au choix du signe :

$$\frac{d^{(2i+1)} f(x)}{dx^{2i+1}} = \mp \frac{1}{2\pi} \int_a^b f(\alpha)\, d\alpha \int_{-\infty}^{+\infty} p^{2i+1} \sin p(x-\alpha)\, dp.$$

(Fourier 1822)

La plus belle nuit de ma vie

J'ai toujours appelé cette nuit de découverte ma nuit merveilleuse, ou la plus belle nuit de ma vie. Dans ma jeunesse, j'avais souvent des insomnies de plusieurs heures et ne prenais jamais de somnifères. Je restais dans mon lit, lumière éteinte, et faisais souvent, évidemment sans rien écrire, des mathématiques. Mon énergie inventive était décuplée, j'avançais avec rapidité sans ressentir de fatigue. J'étais alors totalement libre, sans aucun des freins qu'imposent la réalité du jour et l'écriture. Après quelques heures, la lassitude survenait quand même, surtout si une difficulté mathématique se présentait obstinément. Alors je m'arrêtais et dormais jusqu'au matin. J'étais fatigué tout le jour suivant, mais heureux ; il me fallait souvent plusieurs jours pour tout remettre en ordre. Cette fois-là, j'étais sûr de moi et plein d'exaltation. Dans ce genre de circonstance, je ne perdais pas de temps pour tout expliquer par le menu à Cartan qui, comme je l'ai dit, habitait à côté. Il était lui-même enthousiasmé : « Bon, voilà que tu viens de résoudre toutes les difficultés de la dérivation. Désormais, plus de fonctions sans dérivées », me dit-il. Si une fonction est sans dérivée (Weierstrass), c'est qu'elle a des dérivées qui sont des opérateurs mais ne sont pas des fonctions.

(Schwartz 1997)

Fourier on infinity

417.

La démonstration précédente suppose la notion des quantités infinies, telle qu'elle a toujours été admise par les géomètres. Il serait facile de présenter la même démonstration sous une autre forme en examinant les changements qui résultent de l'accroissement continuel du facteur p sous le signe $\sin p(\alpha - x)$. Ces considérations sont trop connues pour qu'il soit nécessaire de les rappeler.

Analytic symbols as representing physical phenomena

428.

Nous terminerons ici cette Section, dont l'objet appartient presque entièrement à l'Analyse. Les intégrales que nous avons obtenues ne sont point seulement des expressions générales qui satisfont aux équations différentielles : elles représentent de la manière la plus distincte l'effet naturel, qui est l'objet de la question. C'est cette condition principale que nous avons eue toujours en vue, et sans laquelle les résultats du calcul ne nous paraîtraient que des transformations inutiles. Lorsque cette condition est remplie, l'intégrale est, à proprement parler, *l'équation du phénomène*; elle en exprime clairement le caractère et le progrès, de même que l'équation finie d'une ligne ou d'une surface courbe fait connaître toutes les propriétés de ces figures. Pour découvrir ces solutions, nous ne considérons point une seule forme de l'intégrale; nous cherchons à obtenir immédiatement celle qui est propre à la question. C'est ainsi que l'intégrale qui exprime le mouvement de la chaleur dans une sphère d'un rayon donné est très différente de celle qui exprime ce mouvement dans un corps cylindrique, ou même dans une sphère d'un rayon supposé infini. Or chacune de ces intégrales a une forme déterminée, qui ne peut pas être suppléée par une autre. Il est nécessaire d'en faire usage si l'on veut connaître la distribution de la chaleur dans le corps dont il s'agit. En général, on ne pourrait apporter aucun changement dans la forme de nos solutions sans leur faire perdre leur caractère essentiel, qui est de représenter les phénomènes.

(Fourier 1822)

Translation:

We will conclude here this Section, whose subject belongs almost entirely to Analysis. The integrals we have obtained are not merely general expressions that satisfy the differential equations: they represent, in the most distinct manner, the natural effect that is the object of the question. This is the principal condition we have always kept in view, and without it the results of the calculation would appear to us only as useless transformations. When this condition is fulfilled, the integral is, strictly speaking, *the equation of the phenomenon*; it clearly expresses its character and its progression, just as the finite equation of a line or curved surface makes known all the properties of those figures.

To discover these solutions, we do not consider only a single form of the integral; we seek to obtain immediately the one that is suited to the question. Thus the integral that expresses the movement of heat in a sphere of given radius is very different from the one that expresses this movement in a cylindrical body, or even in a sphere of supposedly infinite radius. Now each of these integrals has a determined form, which cannot be replaced by another. It is necessary to make use of this form if one wants to know the distribution of heat in the body in question. In general, one could not introduce any change into the form of our solutions without causing them to lose their essential character, which is to represent the phenomena.